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Three short distance structures from quantum algebras

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Abstract

We review known and we present new results on three types of short distance structures of observables which typically appear in studies of quantum group related algebras. In particular, one of the short distance structures is shown to suggest a new mechanism for the introduction of internal symmetries.

1 Introduction

In studies of quantum group related associative algebras, see e.g. [1]-[9], the commutation relation

$$ab - qba = 0, \qquad , \qquad q \in \mathbf{C} \tag{1}$$

is among the most typical, together with the inhomogenous relation:

$$\tilde{a}\tilde{b} - q'\tilde{b}\tilde{a} = q'', \qquad\qquad q', q'' \in \mathbf{C} \tag{2}$$

As pointed out e.g. in [7], for $q \neq 1$ Eqs.1 and 2 can be transformed into another e.g. through

$$b = \tilde{b}, \qquad a = \tilde{a} + \tilde{b}q''(q-1)^{-1}, \qquad q = q' \tag{3}$$

In the following we will focus on Eq.2.

For quantum mechanical applications it is crucial to identify a canonical pair of hermitean operators, say $\mathbf{x}$ and $\mathbf{p}$ that play the role of observables with real expectation values. With the development of a generalised, noncommutative linear algebra (see e.g. [8] for the braided case) and ultimately functional analysis it may prove possible and fruitful to generalise the very concept of hermiticity. For the time being, however,

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we will have to choose observables which are hermitean in the conventional sense, i.e. when represented on a separable complex Hilbert space. Any choice of observables will be characterised by a choice of commutation relations. At least on a formal algebraic level, many choices of commutation relations can be transformed into another, as e.g. the example of Eqs.1,2,3 shows. We also recall that all separable complex Hilbert spaces are isomorphic. The deeper reason why choices of observables with different commutation relations can, and indeed must exhibit new phsical features, such as a modified short distance structure, is the fact that transformations which change commutation relations cannot be implemented unitarily, as is not difficult to verify.

The first ansatz for an identification of observables is to try to identify $\tilde{a}$ and $\tilde{b}$ directly with observables $\mathbf{x}$ and $\mathbf{p}$, obeying

$$\mathbf{x}\mathbf{p} - q\mathbf{p}\mathbf{x} = i\hbar \tag{4}$$

We will consider this approach only briefly. In this case, if e.g. $\mathbf{p} = \mathbf{p}^\dagger$ is chosen then $\mathbf{x} = \mathbf{x}^\dagger$ is not consistent with Eq.4 for $q \neq 1$, so that an alternative position observable needs to be defined. These studies have been carried out in detail, see e.g. [9]. One of the main results concerning the resulting short distance structure has been that both, position and momentum space become discretised (with exponential spacing).

Let us now turn to a different identification of observables in Eq.2, which will yield the second and third short distance structure which we will here consider. The ansatz is to take as $\tilde{a}$ and $\tilde{b}$ two mutually adjoint operators a and $a^\dagger$ obeying (this a is not to be confused with the a of Eq.1)

$$a a^\dagger - q a^\dagger a = 1, \qquad\qquad q \in \mathbf{R} \tag{5}$$

Observables $\mathbf{x}$ and $\mathbf{p}$ are then identified as usual as the hermitean and anti-hermitean parts of a

$$\mathbf{x} = L(a + a^\dagger), \qquad\qquad \mathbf{p} = iK(a - a^\dagger) \tag{6}$$

where L and K carry units of length and momentum respectively. The involution defined on the generators as $\mathbf{x} = \mathbf{x}^\dagger, \mathbf{p} = \mathbf{p}^\dagger$ is consistent, and Eq.5 translates into

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + (q-1)(\mathbf{x}^2/4L^2 + \mathbf{p}^2/4K^2)) \tag{7}$$

with $KL = \hbar(q+1)/4$. This approach (also for the more general case of $U_q(n)$ comodule algebras) has been introduced in [15, 17], where also the corresponding short distance structure has been analysed: For $q > 1$ both the uncertainty in position and in momentum are separately finitely bounded from below. The main features can also be studied in the simpler case of only one correction term on the RHS of the commutation relation:

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \beta \mathbf{p}^2) \tag{8}$$

The two cases $\beta > 0$ and $\beta < 0$ (corresponding to $q > 1$ and $q < 1$) lead to very different short distance structures. The case of $\beta > 0$ has been studied in considerable detail. The uncertainty relation then reads

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + ...) \tag{9}$$

In the context of string theory and quantum gravity generalised uncertainty relations and in particular relations of this type have been discussed, see e.g. [10]-[26]. The uncertainty relation then expresses the presence of a natural ultraviolet cutoff as expected in the region of the Planck scale. Technically, for all states $|\psi\rangle$ in a representation of the commutation relations there holds $\Delta x_{|\psi\rangle} \geq \Delta x_0 = \hbar\sqrt{\beta}$. We will later return to the case $\beta > 0$.

Let us now begin with the analysis of the case $\beta < 0$, which is new. The uncertainty relation in this case yields for the minimal position uncertainty $\Delta x_0 = 0$, as usual. Taking the trace on both sides of Eq.8 shows that finite dimensional representations are no longer excluded. Indeed, there now exist even one-dimensional representations with $\mathbf{x}$ represented as some arbitrary number and $\mathbf{p}$ represented as $\pm|\beta|^{-1/2}$. All finite dimensional representations reduce to direct sums of these: In the $\mathbf{p}$ eigenbasis $\mathbf{p}_{ij} = \mathbf{p}_i\delta_{ij}$ and the commutation relations, $\mathbf{x}_{rs}(\mathbf{p}_s - \mathbf{p}_r) = i\hbar\delta_{rs}(1+\beta\mathbf{p}_r^2)$ yield $\mathbf{p}_r = \pm|\beta|^{-1/2}$, thus $([\mathbf{x},\mathbf{p}])_{rs} = 0$, so that $\mathbf{x}$ is diagonalisable simultaneously with $\mathbf{p}$, and we obtain $\mathbf{p}_{rs} = \mathrm{diag}(p_1, p_2, ..., p_n)$ and $\mathbf{x}_{rs} = \mathrm{diag}(x_1, x_2, ..., x_n)$ with $p_i \in \{-|\beta|^{-1/2}, |\beta|^{-1/2}\}$ and $x_i \in \mathbf{R}$. There also appear new features in infinite dimensional representations. Consider the spectral representation of $\mathbf{p}$:

$$\mathbf{p}.\psi(\lambda) \;=\; \lambda\psi(\lambda) \tag{10}$$

$$\mathbf{x}.\psi(\lambda) \;=\; i\hbar\left(\frac{d}{d\lambda} + \beta\lambda\frac{d}{d\lambda}\lambda\right)\psi(\lambda) \tag{11}$$

$$\langle\psi_1|\psi_2\rangle \;=\; \int_I d\lambda\ \psi_1^*(\lambda)\psi_2(\lambda) \tag{12}$$

The family of operators G defined through the integral kernel $(a, b \in \mathbf{C})$

$$G(\lambda, \lambda') = \left(a\ \Theta(\lambda - |\beta|^{-1/2}) + b\ \Theta(\lambda + |\beta|^{-1/2})\right)\delta(\lambda - \lambda') \tag{13}$$

commute with both $\mathbf{x}$ and $\mathbf{p}$. Each G is diagonal and constant apart from two steps where it cuts momentum space, and with it the representation, into three unitarily inequivalent parts. The representation which has the proper limit as $\beta \to 0$ is given by Eqs.10-12 with the integration interval $I := I_c = [-|\beta|^{-1/2}, |\beta|^{-1/2}]$. Thus, $\mathbf{p}$ becomes a bounded self-adjoint operator. Let us calculate the defect indices of $\mathbf{x}$ in this representation, i.e. the dimensions of the kernels of $(\mathbf{x}^* \pm i)$. To this end we check for square integrable solutions to

$$i\left(\partial_\lambda - |\beta|(\lambda^2\partial_\lambda + \lambda)\right)\psi_\xi(\lambda) = \xi\psi_\xi(\lambda) \tag{14}$$

with $\xi = \pm i$ (from now on we set $\hbar = 1$). Eq.14 is solved by

$$\psi_\xi(\lambda) = \langle\lambda|\xi\rangle = N \left(1 - |\beta|\lambda^2\right)^{-1/2} \left(1 - \sqrt{|\beta|}\lambda\right)^{i\xi/2} \left(1 + \sqrt{|\beta|}\lambda\right)^{-i\xi/2} \tag{15}$$

which are non-square integrable on I_c for all $\xi \in \mathbf{C}$, in particular also for $\xi = \pm i$. Thus, the defect indices are $(0, 0)$, i.e. $\mathbf{x}$ is still essentially self-adjoint with a unique spectral representation (compare with the well known fact that the operator $i\partial_\lambda$ which ordinarily represents $\mathbf{x}$ on momentum space has defect indices (1,1) on the intervall). The position eigenfunctions are given by Eq.15 for real ξ. With the continuum normalisation $N = (2\pi)^{-1/2}$ it is not difficult to verify orthonormalisation and completeness:

$$\int_{-|\beta|^{-1/2}}^{|\beta|^{-1/2}} d\lambda \ \langle\xi|\lambda\rangle\langle\lambda|\xi'\rangle = \delta(\xi - \xi') \tag{16}$$

$$\int_{-\infty}^{\infty} d\xi \ \langle\lambda|\xi\rangle\langle\xi|\lambda'\rangle = \delta(\lambda - \lambda') \tag{17}$$

The generalised Fourier factor given in Eq.15 yields $\psi(\xi) = \int_{I_c} d\lambda\langle\xi|\lambda\rangle\psi(\lambda)$ for the transformation which maps momentum space wave functions $\psi(\lambda) = \langle\lambda|\psi\rangle$ to position space wave functions $\psi(\xi) = \langle\xi|\psi\rangle$. To summarise in the case of our second short distance structure, the case of $\beta < 0$, we have found the presence of a continuous position spectrum while, unexpectedly, momentum space becomes bounded.

Let us now turn to the case $\beta > 0$ and our third short distance structure. As is well known, and as is easily derived from Eq.9, the position resolution Δx now becomes finitely bounded from below: $\Delta x_0 = \sqrt{\beta}$. To be precise, for all normalised vectors $|\psi\rangle$ in a domain D on which the commutation relations hold the position uncertainty obeys $\Delta x_{|\psi\rangle} = \langle\psi|(\mathbf{x} - \langle\psi|\mathbf{x}|\psi\rangle)^2|\psi\rangle^{1/2} \geq \Delta x_0 = \sqrt{\beta}$. A convenient representation is given by Eqs.10-12 with $I = \mathbf{R}$. On any dense domain D in a Hilbert space H on which the commutation relations hold the position operator can only be symmetric but not self-adjoint, as diagonalisability is excluded by the uncertainty relation (eigenvectors to an observable automatically have vanishing uncertainty in this observable). This also excludes the possibility of finite dimensional representations of the commutation relations (since in these symmetry and self-adjointness coincide), as could of course also be seen by taking the trace of both sides of Eq.8. The underlying functional analytic structure was first discussed in [15, 17].

We will now discuss further physical implications of this short distance structure, related to internal symmetries. Our aim is to show that the unobservability of localisation beyond the minimal uncertainty Δx_0 represents an internal symmetry where degrees of freedom which correspond to small scale structure beyond the Planck scale turn into internal degrees of freedom.

Consider a $*$-representation (such as given by Eqs.10-12) of the commutation relation Eq.8 with $\beta > 0$ on a maximal dense domain D in a Hilbert space H. Then

$\mathbf{x}$ is merely symmetric, i.e. D is smaller than the domain $D_{\mathbf{x}^*}$ of the adjoint operator $\mathbf{x}^*$ (which is not symmetric). The deficiency spaces L_+, L_-, i.e. the spaces spanned by eigenvectors of $\mathbf{x}^*$ with eigenvalues $+i$ and $-i$ are one-dimensional, i.e. the deficiency indices are (1,1).

Thus, there exists a set of self-adjoint extensions of $\mathbf{x}$ which is in one-to-one correspondence with the set of unitary transformations $\tilde{U} : L_+ \to L_-$. We recall that, by the usual procedure, each $\tilde{U}$ defines a unitary extension of the Cayley transform of $\mathbf{x}$, with the inverse Cayley transform then defining a self adjoint extension of $\mathbf{x}$. On the eigenvalues, Cayley transforms are Möbius transforms.

The $\tilde{U}$ differ exactly by the set G of unitary transformations $U : L^+ \to L^+$, which we may here call the 'local group' G. Thus, the set of self-adjoint extensions $\{\mathbf{x}_\alpha\}$ forms a representation of the local group, where α which labels the self-adjoint extensions is a vector in the fundamental representation of G. The local group also acts on the set of spectra $\{\sigma_\alpha\}$ of the $\mathbf{x}_\alpha$. Let us denote the eigenvalues of the self-adjoint extension $\mathbf{x}_\alpha$ by $v_\alpha(r)$. Then, for any fixed r, we obtain an orbit $O(r) := \{v_{U.\alpha} | U \in G\}$ of eigenvalues under the action of G.

For the case of Eq.8 the scalar product of eigenvectors of $\mathbf{x}^*$ has been calculated in [21]:

$$\langle \xi | \xi' \rangle = \frac{2\sqrt{\beta}}{\pi(\xi - \xi')} \sin\left(\frac{\xi - \xi'}{2\sqrt{\beta}}\pi\right) \tag{18}$$

From its zeros we can read off the family of discrete spectra of the self-adjoint extensions:

$$\sigma_\alpha = \left\{ v_\alpha(r) = (2r + s/\pi)\sqrt{\beta} \mid r \in \mathrm{N} \right\} \qquad \text{where} \qquad \alpha = e^{is} \quad \text{with} \quad s \in [0, 2\pi[\tag{19}$$

The spectra are equidistant and self-adjoint extensions differ by a shift of their lattice of eigenvalues. The local group is here the group of translations of the lattices of eigenvalues. Due to the periodicity of the lattice this group is S^1, or $U(1)$. This had to be expected since in this case L_+ is one-dimensional and the self-adjoint extensions therefore form a representation of the local group U(1).

Each choice of self-adjoint extension of the position operators therefore corresponds to a choice of lattice on which the physics takes place. However, the commutation relations also imply that the smallest uncertainty in positions becomes finite and large enough so that the actual choice of lattice cannot be resolved. Technically, all self adjoint extensions of $\mathbf{x}$ coincide when restricted to a domain D on which the commutation relations hold.

If, therefore, with a physical state $|\psi\rangle \in D$ also some vector α is specified, as a choice of self-adjoint extension, the action should be invariant. We therefore arrive at a global symmetry principle where the additional information given by the 'isospinor' α can be interpreted. Assume that the state of a particle is projected onto a state of maximal localisation ($\Delta x = \Delta x_0$) with position expectation ξ. Specifying α is to

specify one point in the orbit of the eigenvalue ξ under the action of the local group. As a convention one can specify that this is where the maximally localised particle (or one point of it if it is viewed as an extended particle) is said to "actually" sit. This is consistent because the radius of the orbits of the eigenvalues is $\sqrt{\beta} = \Delta x_0$ i.e. of the size of the finite minimal uncertainty Δx_0, so that all these conventions, differing only by the action of the local group, cannot be distinguished observationally. For example, the pointwise multiplication of fields as discussed e.g. in [21] can be reformulated in terms of a choice of position eigenbasis, rather than the set of maximally localised fields. The gauge principle is that the action is invariant under the local group. We remark that the proof of ultraviolet regularity still goes through since not only the fields of maximal localisation, but also the position eigenfields are normalisable. The 'local' group may then also be taken to act locally, i.e. we consider $|\psi\rangle \in D \otimes L_+$. It is unobservable whether one specifies one self-adjoint extension's lattice here and another's there, as long as the parallel transport of α is consistently defined. At large scales this should turn into the ordinary local gauge principle.

There is therefore a possibility that internal symmetry spaces arise as deficiency spaces of position operators. Introducing $\Delta x_0 > 0$ the infinite dimensional Hilbert space of fields develops special dimensions that correspond to degrees of freedom that describe localisation beyond what can be resolved, and which can therefore be viewed as internal degrees of freedom. In the case of one dimension our studies yield the simple intuitive picture that certain corrections to the uncertainty relations lead to physics on a whole set of possible lattices, while the choice of any particular lattice from the set cannot be resolved and does therefore correspond to an internal degree of freedom. This may be a new mechanism, or it could be related to the Kaluza Klein mechanism. It should be very interesting to explore possible connections to string theory.

In particular, it should be interesting to explore a possible connection to an observation by Faddeev: In [27] he noted a simple mechanism by which in the process of compactification or discretisation one degree of freedom with infinitesimal generators of $\mathbf{x}$ and $\mathbf{p}$ with $[\mathbf{x}, \mathbf{p}] = i\hbar$ can turn into two degrees of freedom of ξ_1, π_1 and ξ_2, π_2 where the ξ_i and π_i are now finite translations and boosts: $\xi_i = \exp(ia_i\mathbf{x}),\ \pi_i = \exp(b_i\mathbf{p})$, with appropriately chosen parameters a_i and b_i, yielding a modular group action. These degrees of freedom would here correspond to the (external) degree of freedom which measures distances in units of the fundamental length, and the (internal) degree of freedom which is related to the position of the lattice of eigenvalues, technically analogous to the pair of winding and ordinary modes degrees of freedom.

Finally, let us remark that general commutation relations, for all types of short distance structure, as long as $\mathbf{x}_i = \mathbf{x}_i^\dagger$ and $\mathbf{p}_i = \mathbf{p}_i^\dagger$, can be introduced into the quantum field theoretical path integral. The main observation is that the functional analysis of representations of the commutation relations on wave functions extends to representations on fields. For early considerations on the representation of the $\mathbf{x}, \mathbf{p}$

commutation relations on the space of fields which is formally being summed over in the field theoretic path integral see e.g. [28]. In particular, $\Delta x_0 = \hbar\sqrt{\beta} > 0$, in spite of being an ensemble-cutoff rather than an individual-case-cutoff, has been shown to regularise the ultraviolet in euclidean field theory. The issue of regularisation through $\Delta x_0 > 0$ has been studied extensively in [29]-[33].

Our analysis concerning internal symmetries covered the case of one space-like dimension with a positive correction term to the commutation relations. Due to the indefinite Minkowski signature the temporal coordinate can be expected to come with correction terms with a negative sign i.e. with a short distance structure of the second type ($\beta < 0$) which we here considered. Our study above of this case indicates that this direction, i.e. $\mathbf{x}_0$, should have a unique self-adjoint extension and that it therefore may not contribute to the internal symmetries. The generic case of d dimensions with Minkowski signature and possibly higher order correction terms will require careful investigation. It is of course possible that more than the here covered three short distance structures may appear.

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